A Simple Exploration on Symplectic Spaces and Two Especial Linear Transformations on Them

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May 5, 2021

Abstract

Similar to two kinds of inner product spaces, Euclidean spaces and unitary spaces, we would like to define a nondegenerate skew-symmetric bilinear forms as an inner product on a finite-dimensional linear space over a number field of characteristic not 2, in order to derive symplectic spaces and two kinds of especial linear transformations on them, and then explore some simple properties of them.

Contents

1 Introduction

Symplectic Spaces are a linear space with a skew-symmetric inner product, here we imitate what we have done in inner products, and try to do some explorations on these unique spaces in a similar way.

In the first section, we will define such symplectic spaces, then we will give the isomorphism of them, and do some simple researches on their substructures.

In the second section, developed the theory of symplectic spaces, we will explore two kinds of especial linear transformations on it. One preserves the inner product, and the other is self-adjoint. This section contains some simple property of them, as well as the characteristic of their matrix, and discovery of a group structure behind the former one and so on.

Ahead of discussion, we'd like to emphasize it that all the discussion is under finite-dimensional spaces V and number fields $\mathbb{K}$ of characteristic not 2.

2 Symplectic Forms

We will concisely restate some properties of skew-symmetric bilinear forms and nondegenerate ones which are also called symplectic forms in this section, in order to build symplectic spaces more conveniently later.

2.1 Skew-symmetric Bilinear Forms

Different from the inner product on Euclidean spaces, we change the symmetry inner product into skew-symmetric one.

Definition 2.1. *Suppose V is a linear space over $\mathbb{K}$. Then the* ***skew-symmetric bilinear form*** *is a bilinear form $(\cdot,\cdot)\colon V \times V \to \mathbb{K}$ fulfilling*

$$\forall \alpha, \beta \in V, (\alpha, \beta) = -(\beta, \alpha) \tag{1}$$

Generally, skew-symmetric bilinear forms can be described by following alternation property.

Property 2.1. $(\cdot,\cdot)$ *is a skew-symmetric bilinear form over $\mathbb{K}$ if and only if*

$$(\alpha, \alpha) = 0 \ (\forall \alpha \in V). \tag{2}$$

Proof:
(=>)
Suppose $(\cdot,\cdot)$ is a skew-symmetric bilinear form. Then $(\alpha, \alpha) = -(\alpha, \alpha)$.
$\implies 2(\alpha, \alpha) = 0 \implies (\alpha, \alpha) = 0$.
(<=)
Suppose $(\alpha, \alpha) = 0 (\forall \alpha \in V)$.
Then $(\alpha + \beta, \alpha + \beta) = (\alpha, \alpha) + (\beta, \beta) + (\alpha, \beta) + (\beta, \alpha) = (\alpha, \beta) + (\beta, \alpha) = 0$.
Thus $(\alpha, \beta) = -(\beta, \alpha)$. □

Through the process below, we can find a standard form of the skew-symmetric bilinear form's metric matrix.

Theorem 2.1. *Suppose $(\cdot,\cdot) : V \times V \to \mathbb{K}$ is a skew-symmetric bilinear form. Then there exists a basis to make its metric matrix shaped like*

$$\begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & & & & & & \\ & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & & & & & \\ & & \ddots & & & & \\ & & & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}. \tag{3}$$

Proof:
It's trivial when $(\cdot, \cdot) \equiv 0$, so just consider the situation where $(\cdot, \cdot) \not\equiv 0$.
If $\dim V = 1$, it can only be that $(\cdot, \cdot) \equiv 0$.
Make an inductive hypothesis that it's true for all $\dim V < n$. Then consider the situation where $\dim V = n$,
$\exists \alpha, \beta \in V \ s.t. \ (\alpha, \beta) = d \neq 0$ (otherwise $(\cdot, \cdot) \equiv 0$),
and then substitute α with $\frac{\alpha}{d}$.
Obviously, $(\alpha, \beta) = 1, (\beta, \alpha) = -1$ on this condition.
Assert α, β (otherwise $(\alpha, \beta) = 0$) is linearly independent, and expend it to be a basis of V: $\alpha, \beta, \gamma, \eta \cdots$

Perform the following process of orthogonalization blow to make $\gamma, \eta \cdots$ orthogonal to α, β:
$\gamma' = \gamma + (\beta, \gamma)\alpha - (\alpha, \gamma)\beta$,
$\eta' = \eta + (\beta, \eta)\alpha - (\alpha, \eta)\beta$,
$\ldots$
We get the basis suitable for our requirements: $\alpha, \beta, \gamma', \eta' \cdots$.
Let a subspace $W = \operatorname{span}\{\gamma', \eta' \cdots\}$.
The restriction map $(\cdot, \cdot)|_W$ is also a skew-symmetric bilinear form.
As $\dim W < \dim V$, based on the inductive hypothesis,
there exists a basis of W: $\gamma'', \eta'' \cdots$, under which the metric matrix is in that shape.
Thus choose the basis $\alpha, \beta, \gamma'', \eta'' \cdots$ of V, under which the metric matrix is also in that shape, as desired. □

Remark. *Denote this process above by the **symplectically orthonormalizing process**.*

Corollary 2.1. *Suppose $(\cdot, \cdot)$ is a skew-symmetric bilinear form, M is the metric matrix of it. Then* $\operatorname{rank}(\cdot, \cdot) = \operatorname{rank} M$ *is even.*

2.2 Nondegenerate Skew-symmetric Bilinear Forms

To avoid the existence of a vector being orthogonal to any other vectors, we need do define nondegenerate forms.

Definition 2.2. *Suppose V is a linear space, a skew-symmetric bilinear form is nondegenerate if*

$$\forall \beta \in V, (\alpha, \beta) = 0 \implies \alpha = 0. \tag{4}$$

Property 2.2. *Suppose V is a linear space, a skew-symmetric bilinear form is nondegenerate if and only if:*

$$\forall \alpha \in V, (\alpha, \beta) = 0 \implies \beta = 0. \tag{5}$$

Proof:

$$\begin{aligned} &(\forall \beta \in V, (\alpha, \beta) = 0 \implies \alpha = 0) \\ \iff &(\forall \beta \in V, (\beta, \alpha) = -(\alpha, \beta) = 0 \implies \alpha = 0). \end{aligned}$$

□

Definition 2.3. *Suppose V is a linear space with dimension $2k (k \in \mathbb{N}_+)$. Then a symplectic form on V is a nondegenerate skew-symmetric bilinear forms.*

Property 2.3. *Suppose V is a linear space over $\mathbb{K}$, $(\cdot, \cdot)$ is a symplectic form if and only if there exists a basis under which the metric matrix is*

$$\begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & & & \\ & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & & \\ & & \ddots & \\ & & & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \end{pmatrix}. \tag{6}$$

Proof:
(=>)
Choose a basis of V, and then do the symplectically orthonormalizing process on it, and we get a basis $\eta_1, \eta_2 \cdots \eta_n$.

Because $(\cdot,\cdot)$ is nondegenerate,
assert that for all $\eta_i (i \in \{1, \cdots, n\})$, there always exists another $\eta_j (j \in \{1, \cdots, n\}, j \neq i)$, such that $(\eta_i, \eta_j) \neq 0$.
If not so, for all $\eta_j (j \in \{1, \cdots, n\})$, we can conclude $\forall \eta_i (i \in \{1, \cdots, n\}), (\eta_i, \eta j) = 0$,
$\implies \forall \alpha \in V, (\eta_i, \alpha) = 0$.
Since $\eta_i \neq 0$, it's a contradiction to $(\cdot,\cdot)$ is nondegenerate.
So there is no row or column being all zeros in the metric matrix, and the matrix is what we want, as desired.
(<=)
Suppose there exists a basis $\eta_1, \eta_2, \cdots, \eta_{2k}$ under which the metric matrix is like this.
Then $\forall \alpha \neq 0 \in V$, assume $\alpha = \sum\limits_{i=1}^{2k} p_i \eta_i (p_i \in \mathbb{K})$, and $p_i (i \in \{1, 2, \cdots, 2k\})$ are not all zero.
Without loss of generality, suppose $p_1 \neq 0$. Then $(\eta_2, \alpha) = p_1 \neq 0$, and thus $(\cdot, \cdot)$ is nondegenerate.
Therefore, $(\cdot, \cdot)$ is a symplectic form. $\square$

Corollary 2.2. *Suppose $(\cdot,\cdot)$ is a symplectic form on linear space V. Then the metric matrix of $(\cdot, \cdot)$ must be full rank and* $\dim V$ *must be even.*

3 Symplectic Spaces

3.1 Symplectic Spaces

Define a symplectic form as an inner product onto a linear space, deriving the symplectic space.

Definition 3.1. *A symplectic space* $(V, (\cdot, \cdot))$ *is a linear space* V *over* $\mathbb{K}$*, and* $\dim V = 2k(k \in \mathbb{N}_+)$*, with a symplectic form* $(\cdot, \cdot)$ *on it.*

We can give a orthogonal-like definition in a symplectic space.

Definition 3.2. *Suppose* V *is a symplectic space. Then* $\forall \alpha, \beta \in V$*,* α *and* β *are symplectically orthogonal if and only if* $(\alpha, \beta) = (\beta, \alpha) = 0$*.*

Property 3.1. *Suppose* V *is symplectic space. Then all the vectors in* V *are isotropic or symplectically orthogonal to itself i.e.*

$$\forall \alpha \in V,\ (\alpha, \alpha) = 0. \tag{7}$$

According to the property of its inner product, all the vectors are isotropic in the space, so we don't define the length and the angle here yet.

Similar to the orthonormal basis in inner product spaces, we can find an analogous basis through theorem 2.1 to make the metric matrix be a standard form.

Property 3.2. *Suppose* V *is a symplectic space with dimension* $2k(k \in \mathbb{N}_+)$*. Then there exists a basis* $\eta_1, \eta_2 \cdots \eta_{2k}$*, under which the metric matrix of the inner product is*

$$\overline{S_{2k}} = \begin{pmatrix} S_2 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_2 \end{pmatrix} (S_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}). \tag{8}$$

Proof: We can draw this conclusion directly from property 2.3, as desired. □

Remark. *Denote* $\eta_1, \eta_2 \cdots \eta_{2k}$ *by a* ***primary symplectically orthonormal basis****, and* $\overline{S_{2k}}$ *a* $2k \times 2k$ ***primary symplectically congruent matrix****.*

Theorem 3.1. *Suppose* V *is a symplectic space, and* $\dim V = 2k(\in \mathbb{N}_+)$*. Then there exists a basis* $\eta_1, \eta_2 \cdots \eta_{2k}$*, under which the metric matrix of the inner product is*

$$S_{2k} = \begin{pmatrix} & E_k \\ -E_k & \end{pmatrix} \tag{9}$$

Proof:
Choose a primary symplectically orthonormal basis of V: $\eta_1, \xi_1, \eta_2, \xi_2 \cdots \eta_k, \xi_k$.
It's clear that the metric matrix under $\eta_1, \eta_2, \cdots, \eta_k, \xi_1, \xi_2, \cdots \xi_k$ is S_{2k}, as desired. □

Remark. *Denote* $\eta_1, \eta_2, \cdots, \eta_k, \xi_1, \xi_2, \cdots \xi_k$ *by a* ***(secondary) symplectically orthonormal basis****, and* S_{2k} *a* $2k \times 2k$ ***(secondary) symplectically congruent matrix****.*

3.2 Isomorphism of Symplectic Spaces

Here we will illustrate the isomorphism of symplectic spaces, which are isomorphic as linear spaces and preserved the inner product.

Definition 3.3. *Suppose* V *and* V' *are both symplectic spaces. Then they are isomorphic if and only if there exists a linear map (or an isomorphism)* $\mathscr{A} \in \mathrm{Hom}(V, V')$ *fulfilling* *(i)* $\mathscr{A}$ *is bijective;*

(ii) $\forall \alpha, \beta \in V, (\mathscr{A}\alpha, \mathscr{A}\beta) = (\alpha, \beta)$.

Dimensions are actually a complete invariant to describe an isomorphism on symplectic spaces, just like what we've done on inner product spaces.

Theorem 3.2. *Suppose* V *and* V' *are both symplectic spaces. Then they are isomorphic if and only if*

$$\dim V = \dim V'. \tag{10}$$

Proof:
(=>)
This direction is obviously true.
(<=)
Suppose $\dim V = \dim V = 2k (k \in \mathbb{N}_+)$, and then choose a pair of symplectically orthonormal bases from each. Then we get two bases, $\xi_1, \xi_2, \cdots, \xi_{2k}$ from V, and $\eta_1, \eta_2, \cdots, \eta_{2k}$ from V'.
On the one hand, construct a linear map $\mathscr{A}: V \to V'$ satisfying $\mathscr{A}\xi_i = \eta_i (\forall i \in \{1, 2, \cdots, 2k\})$.
It's obvious that $\mathscr{A}$ is both injective and surjective, so $\mathscr{A}$ is bijective.
On the other hand, $\forall \alpha, \beta \in V$, suppose $\alpha = (\xi_1, \xi_2, \cdots, \xi_{2k})X$, and $\beta = (\xi_1, \xi_2, \cdots, \xi_{2k})Y$ $(X, Y \in \mathbb{K}^{2k})$.
It's clear that $\mathscr{A}\alpha = (\eta_1, \eta_2, \cdots, \eta_{2k})X$, and $\mathscr{A}\beta = (\eta_1, \eta_2, \cdots, \eta_{2k})Y$.
Because the metric matrices under these two bases are the same, we get $(\alpha, \beta) = X^T S_{2k} Y = (\mathscr{A}\alpha, \mathscr{A}\beta)$.
Thus $\mathscr{A}$ is an isomorphism, and all the symplectic spaces with the same dimension are always isomorphic.
□

From theorem 3.2, we notice that all the symplectic spaces on $\mathbb{K}$ with dimension $2k (k \in \mathbb{N}_+)$ are isomorphic to $\mathbb{K}^{2k}$.

3.3 Symplectic Subspaces

It's clear that all the subspaces of a symplectic subspace is not a symplectic space. We'd like to give a definition to its symplectic subspace.

Definition 3.4. *Suppose* V *is a symplectic space, and* W *is a subspace of* V*. Then* W *is a symplectic subspace of* V *if* W *is a symplectic space.*

Property 3.3. *Suppose* V *is a symplectic space, and* W *is a subspace of* V*. Then* W *is a symplectic subspace of* V *if and only if the inner product of* V *is nondegenerate on* W*.*

Proof:
This property is obviously true from definition 3.1. □

We want to work out the condition where a subspace can be a symplectic subspace. Ahead of that, let's consider whether we can extend a symplectically orthonormal basis of a symplectic subspaces to the symplectically orthonormal basis of the total space.

Lemma 3.1. *Suppose* V *is a symplectic space with dimension* $2k(k \in \mathbb{N}_+)$*, and* W *is a symplectic subspace with a symplectically orthonormal basis* $\xi_1, \xi_2, \cdots, \xi_r, \eta_1, \eta_2, \cdots, \eta_r (r \in \mathbb{N}_+, r \leq k)$*. Then this basis can be extended to a symplectically orthonormal basis* $\xi_1, \xi_2, \cdots \xi_r, \xi_{r+1}, \cdots, \xi_k, \eta_1, \eta_2, \cdots \eta_r, \eta_{r+1}, \cdots, \eta_k$ *of* V*.*

Proof:
In order to make the proof concise, we rearrange the symplectically orthonormal basis to make it a primary one of W,
i.e. arrange $\xi_1, \xi_2, \cdots, \xi_r, \eta_1, \eta_2, \cdots, \eta_r$ into $\xi_1, \eta_1, \xi_2, \eta_2, \cdots, \xi_r, \eta_r$.
Then we extend it to be a basis of V: $\xi_1, \eta_1, \xi_2, \eta_2, \cdots, \xi_r, \eta_r, {\xi_{r+1}}', {\eta_{r+1}}', \cdots, {\xi_k}', {\eta_k}'$.
Perform the symplectically orthonormalizing process on ${\xi_{r+1}}', {\eta_{r+1}}', \cdots, {\xi_k}', {\eta_k}'$, to make it not only orthogonal to $\xi_1, \eta_1, \xi_2, \eta_2, \cdots, \xi_r, \eta_r$, but itself a symplectically orthonormal basis of $\mathrm{span}\{\xi_1, \eta_1, \xi_2, \eta_2, \cdots, \xi_r, \eta_r\}$.
Thus we can get a primary symplectically orthonormal basis $\xi_1, \eta_1, \xi_2, \eta_2, \cdots, \xi_r, \eta_r, \xi_{r+1}, \eta_{r+1}, \cdots, \xi_k, \eta_k$.
Therefore, $\xi_1, \xi_2, \cdots \xi_r, \xi_{r+1}, \cdots, \xi_k, \eta_1, \eta_2, \cdots \eta_r, \eta_{r+1}, \cdots, \eta_k$ is a symplectically orthonormal basis of V, as desired. □

Then come back to the main problem how can a subspace can be a symplectic subspace.

Theorem 3.3. *Suppose V is a symplectic space with dimension $2k(k \in \mathbb{N}_+)$, and W is a subspace of V. Then W is a symplectic subspace if and only if there exist a symplectically orthonormal basis $\xi_1, \xi_2, \cdots, \xi_k, \eta_1, \eta_2, \cdots, \eta_k$ of V and a substitution $(i_1, i_2, \cdots, i_k) = \sigma(1, 2, \cdots, k)$ such that*

$$W = \mathrm{span}\{\xi_{i_1}, \xi_{i_2}, \cdots, \xi_{i_r}, \eta_{i_1}, \eta_{i_2}, \cdots, \eta_{i_r}\}(r \in \mathbb{N}, r \leq k). \tag{11}$$

Proof:
(<=)
Suppose $\xi_1, \xi_2, \cdots, \xi_k, \eta_1, \eta_2, \cdots, \eta_k$ is a symplectically orthonormal basis of V, a substitution $(i_1, i_2, \cdots, i_k) = \sigma(1, 2, \cdots, k)$, and $W = \mathrm{span}\{\xi_{i_1}, \xi_{i_2}, \cdots, \xi_{i_r}, \eta_{i_1}, \eta_{i_2}, \cdots, \eta_{i_r}\}(r \in \mathbb{N}, r \leq k)$.
Under $\xi_1, \xi_2, \cdots, \xi_k, \eta_1, \eta_2, \cdots, \eta_k$ the metric matrix is S_{2k}.
Choose rows and columns both are $i_1, i_2, \cdots, i_r$ to form a submatrix S_{2r}, which is rightly the metric matrix of W's inner product under the basis $\xi_{i_1}, \xi_{i_2}, \cdots, \xi_{i_r}, \eta_{i_1}, \eta_{i_2}, \cdots, \eta_{i_r}$.
This implies W has a symplectic inner product, and thus W is a symplectic space.
(=>)
Suppose W is a symplectic subspace.
Then W has a symplectically orthonormal basis $\xi_1, \xi_2, \cdots, \xi_r, \eta_1, \eta_2, \cdots, \eta_r (r \in \mathbb{N}, r \leq k)$.
According to lemma 3.1, we can extend it to a symplectically orthonormal basis $\xi_1, \xi_2, \cdots, \xi_k, \eta_1, \eta_2, \cdots, \eta_k$ of V. Then choose $\sigma = e$, completing the proof. □

3.4 Symplectically Orthogonal Subspaces

Similar to orthogonal subspace in Euclidean space, all vectors symplectically orthogonal to all the vectors in a subspace can form a subspace.

Definition 3.5. *Suppose V is a symplectic space and W is a subspace of V, and define a subset $W^\perp := \{\alpha \in V \mid (\alpha, \beta) = 0, \forall \beta \in W\}$.*

Property 3.4. *$W^\perp$ is a subspace of V. Denote it by a symplectically orthogonal space of W.*

Proof:
Suppose V is over $\mathbb{K}$.
It's obvious that $0 \in W^\perp$, so $W^\perp$ is not empty.
As $\forall \alpha, \beta \in W^\perp \implies \forall \gamma \in W, (\alpha + \beta, \gamma) = (\alpha, \gamma) + (\alpha, \beta) = 0$, thus $\alpha + \beta \in W^\perp$.
$\forall \alpha \in W^\perp, k \in \mathbb{K} \implies \forall \gamma \in W, (k\alpha, \gamma) = k(\alpha, \gamma) = 0$.
Therefore, $W^\perp$ is a subspace of V, as desired. □

From the symplectically congruent matrix, if W is spanned by a part of symplectically orthonormal basis, we can see some characteristics of $W^\perp$.

Property 3.5. *Suppose V is a symplectic space with dimension $2k(k \in \mathbb{N}_+)$, $\xi_1, \xi_2, \cdots, \xi_k, \eta_1, \eta_2, \cdots, \eta_k$ is a symplectically orthonormal basis, and $W = \text{span}\{\xi_{i_1}, \xi_{i_2}, \cdots, \xi_{i_r}\}(r \in \mathbb{N}, r \leq k, i_j \in \{1, 2, \cdots, k\}(\forall j \in \{1, 2, \cdots, r\}))$. Then $\text{span}\{\xi_1, \xi_2, \cdots, \xi_k\} \subseteq W^{\perp}$.*

Proof:
$\forall \alpha \in \text{span}\{\xi_1, \xi_2, \cdots, \xi_k\}$, assume $\alpha = \sum_{i=1}^{k} p_i \xi_i$.
As $\forall i, j \in \{1, 2, \cdots, k\}, (\xi_i, \xi_j) = 0$, thus $\forall \beta \in W, (\alpha, \beta) = 0 \implies \alpha \in W^{\perp}$.
Therefore, $\text{span}\{\xi_1, \xi_2, \cdots, \xi_k\} \subseteq W^{\perp}$, as desired. □

Corollary 3.1. *Suppose V is a symplectic space with dimension $2k(k \in \mathbb{N}_+)$, $\xi_1, \xi_2, \cdots, \xi_k, \eta_1, \eta_2, \cdots, \eta_k$ is a symplectically orthonormal basis, and $W = \text{span}\{\xi_{i_1}, \xi_{i_2}, \cdots, \xi_{i_r}\}(r \in \mathbb{N}, r \leq k, i_j \in \{1, 2, \cdots, k\}(\forall j \in \{1, 2, \cdots, r\}))$. Then $W \subseteq W^{\perp}$.*

Proof:
From property 3.5, $W \subseteq \text{span}\{\xi_1, \xi_2, \cdots, \xi_k\} \subseteq W^{\perp}$, as desired. □

Remark. *On the conditions in corollary 3.1, noticed that $W = W^{\perp}$ when $W = \text{span}\{\xi_1, \xi_2, \cdots, \xi_k\}$.*

Noticed that property 3.5 and corollary 3.1 are right as well when $W = \text{span}\{\xi_{i_1}, \xi_{i_2}, \cdots, \xi_{i_r}\}(r \in \mathbb{N}, r \leq k, i_j \in \{1, 2, \cdots, k\}(\forall j \in \{1, 2, \cdots, r\}))$, we're showed that $W + W^{\perp}$ is probably not a direct sum.

This directs us to search for the conditions where W and $W^{\perp}$ can form a direct sum, or further $V = W \oplus W^{\perp}$. We surmise and then prove that the following thing is true.

Theorem 3.4. *Suppose V is a symplectic space, and W is a subspace of V. Then $W + W^{\perp}$ is a direct sum if and only if W is a symplectic subspace.*

Proof:
(<=)
Suppose W is a symplectic subspace of V.
From theorem 3.3, there exist a symplectically orthonormal basis $\xi_1, \xi_2, \cdots, \xi_k, \eta_1, \eta_2, \cdots, \eta_k$ of V and a substitution $(i_1, i_2, \cdots, i_k) = \sigma(1, 2, \cdots, k)$ such that $W = \text{span}\{\xi_{i_1}, \xi_{i_2}, \cdots, \xi_{i_r}, \eta_{i_1}, \eta_{i_2}, \cdots, \eta_{i_r}\}(r \in \mathbb{N}, r \leq k)$.
It's obvious that $\xi_{i_1}, \xi_{i_2}, \cdots, \xi_{i_r}, \eta_{i_1}, \eta_{i_2}, \cdots, \eta_{i_r}$ is a symplectically orthonormal basis of W.
$\forall t \in \{1, 2, \cdots, r\}, (\xi_{i_t}, \eta_{i_t}) \neq 0 \implies \xi_{i_t} \notin W^{\perp}$.
Similarly, $\forall t \in \{1, 2, \cdots, r\}, \eta_{i_t} \notin W^{\perp}$.
Therefore, $W \cap W^{\perp} = 0 \implies W \oplus W^{\perp}$ is valid.
(=>)
On the contrary, suppose W is not a symplectic subspace. Then from property 3.3, we know that its inner product is degenerate.
Therefore, there exists $\alpha \neq 0 \in W$ such that $\forall \beta \in W, (\alpha, \beta) = 0 \implies \alpha \in W^{\perp}$,
which implies $W \cap W^{\perp} \neq 0 \implies W + W^{\perp}$ is not a direct sum. A contradictory. □

Therefore, only when W is a symplectic subspace can $W + W^{\perp}$ be a direct sum. The following thing shows if $W \oplus W^{\perp}$, it's rightly the total space.

Corollary 3.2. *Suppose V is a symplectic space, W is a symplectic space of V. Then $V = W \oplus W^{\perp}$.*

Proof:
We can suppose W suitable for the condition in theorem 3.4.
To prove $V = W \oplus W^{\perp}$, we just need to check $W^{\perp} = \text{span}\{\xi_{i_{r+1}}, \xi_{i_{r+2}}, \cdots, \xi_{i_k}, \eta_{i_{r+1}}, \eta_{i_{r+2}}, \cdots, \eta_{i_k}\}$.
$\forall p \in \{r+1, r+2, \cdots, k\}, \forall q \in \{1, 2, \cdots, r\}, (\xi_{i_p}, \xi_{i_q}) = 0, (\xi_{i_p}, \eta_{i_q}) = 0 \implies \xi_{i_p} \in W^{\perp}$.

Similarly, $\forall p \in \{r+1, r+2, \cdots, k\}, \eta_{i_p} \in W^{\perp}$.
Since there are no more vector from the basis $\xi_1, \xi_2, \cdots, \xi_k, \eta_1, \eta_2, \cdots, \eta_k$ of V in $W^{\perp}$, thus $W^{\perp} = \operatorname{span}\{\xi_{i_{r+1}}, \xi_{i_{r+2}}, \cdots, \xi_{i_k}, \eta_{i_{r+1}}, \eta_{i_{r+2}}, \cdots, \eta_{i_k}\}$, which implies $V = W \oplus W^{\perp}$. □

Corollary 3.3. *Suppose V is a symplectic space, W is a symplectic space of V. Then* $\dim V = \dim W + \dim W^{\perp}$.

With these theories before, we can do direct sum decomposition to a vector in V on a symplectic subspace W.

Definition 3.6. *Suppose V is a symplectic space, and W is a symplectic subspace of V. Then $\forall \alpha \in V$, it can be decomposed as $\alpha = \alpha_1 + \alpha_2$ where $\alpha_1 \in W$ and $\alpha_2 \in W^{\perp}$. Denote this decomposition by a symplectically orthogonal decomposition, and α_1 by a symplectically orthogonal projection on W.*

Definition 3.7. *Suppose V is a symplectic space, and W is a symplectic subspace of V. As $\forall \alpha \in V$, it can be decomposed as $\alpha = \alpha_1 + \alpha_2$ where $\alpha_1 \in W$ and $\alpha_2 \in W^{\perp}$, then define a map $\mathscr{P}_W^S(\alpha) := \alpha_1$.*

Property 3.6. *$\mathscr{P}_W^S$ is a linear transformation.*

Proof:
Suppose V is a symplectic space over $\mathbb{K}$, and W is a symplectic subspace of V.
$\forall \alpha, \beta \in V, k \in \mathbb{K}$, assume $\alpha = \alpha_1 + \alpha_2, \beta = \beta_1 + \beta_2$, where $\alpha_1, \beta_1 \in W$ and $\alpha_2, \beta_2 \in W^{\perp}$.
As $k(\alpha_1 + \beta_1) \in W$ and $k(\alpha_2 + \beta_2) \in W^{\perp}$, then $\mathscr{P}_W^S(k(\alpha + \beta)) = \mathscr{P}_W^S(k(\alpha_1 + \beta_1) + k(\alpha_2 + \beta_2)) = \mathscr{P}_W^S(k(\alpha_1 + \beta_1)) = k\alpha_1 + k\beta_1 = k\mathscr{P}_W^S(\alpha) + k\mathscr{P}_W^S(\beta)$.
Thus it's a linear transformation on V, as desired. □

Property 3.7. $(\mathscr{P}_W^S)^2 = \mathscr{P}_W^S$.

Proof:
Since $\mathscr{P}_W^S|_W$ is an identity transformation on W, it's obvious that $(\mathscr{P}_W^S)^2 = \mathscr{P}_W^S$, as desired. □

Considering that almost no geometric metric has been built so far, we don't do further discussions on symplectically orthogonal projection in this paper.

4 Especial Transformations on Symplectic Spaces

In this section, we will discuss two kinds of especial transformations on symplectic spaces. One is symplectic transformations, which preserves the inner product of each pair vectors in a symplectic spaces. The other is symplectically self-adjoint transformations, which is self-adjoint on the space.

4.1 Symplectic Transformations

We'll define a kind of especial linear transformations on symplectic spaces like orthonormal transformations, which preserve the inner product.

Definition 4.1. *Suppose V is a symplectic space. Then $\mathscr{A} \in \mathrm{End}(V)$ is a* ***symplectic transformation*** *if*

$$\forall \alpha, \beta \in V, (\mathscr{A}\alpha, \mathscr{A}\beta) = (\alpha, \beta). \tag{12}$$

Since symplectic transformations preserve the inner product on a symplectic space, it will map a symplectically orthonormal basis to a another symplectically orthonormal one. In addition, the converse proposition is also true.

Property 4.1. *Suppose V is a symplectic space with dimension $2k(k \in \mathbb{N}_+)$ and a symplectically orthonormal basis $\eta_1, \eta_2, \cdots, \eta_{2k}$. Then $\mathscr{A}$ is a symplectic transformation if and only if $\mathscr{A}\eta_1, \mathscr{A}\eta_2, \cdots, \mathscr{A}\eta_{2k}$ is also a symplectically orthonormal basis.*

Proof:
(=>)
As $\mathscr{A}$ is a symplectic transformation, the matrix $((\mathscr{A}\eta_i, \mathscr{A}\eta_j))_{2k\times 2k} = ((\eta_i, \eta_j))_{2k\times 2k}$
So the metric matrix under $\mathscr{A}\eta_1, \mathscr{A}\eta_2, \cdots, \mathscr{A}\eta_{2k}$ is rightly S_{2k}, which implies it's also a symplectically orthonormal basis.
(<=)
Suppose $\mathscr{A}\eta_1, \mathscr{A}\eta_2, \cdots, \mathscr{A}\eta_{2k}$ is a symplectically orthonormal basis.
According to the proof of theorem 3.2, $\mathscr{A}$ is an automorphism (a isomorphism from and to itself) on V, thus it preserves the inner product.
So $\forall \alpha, \beta \in V, (\mathscr{A}\alpha, \mathscr{A}\beta) = (\alpha, \beta)$. □

Remark. *As an identity transformation on V is clearly a symplectic transformation, so this kind of transformations always exists. Denote all the symplectic transformations on V by $\mathrm{SY}(V)$.*

This fact below shows the characteristic of symplectic transformations' matrix under a symplectically orthonormal basis.

Theorem 4.1. *Suppose V is a symplectic space with dimension $2k(k \in \mathbb{N}_+)$. Assume $\mathscr{A} \in \mathrm{End}(V)$, and the metric matrix of $\mathscr{A}$ under a symplectically orthonormal basis is A. Then $\mathscr{A} \in \mathrm{SY}(V)$ if and only if*

$$A^T S_{2k} A = S_{2k}. \tag{13}$$

Proof:
Choose a symplectically orthonormal basis $\eta_1, \eta_2, \cdots, \eta_{2k}$, and suppose $\mathscr{A}(\eta_1, \eta_2, \cdots, \eta_{2k}) = (\eta_1, \eta_2, \cdots, \eta_{2k})A$.
(=>)
Let $A = (X_1, X_2, \cdots, X_{2k})$, and then $\mathscr{A}\eta_i = (\eta_1, \eta_2, \cdots, \eta_{2k})X_i$.
Then we get $(\eta_i, \eta_j) = (\mathscr{A}\eta_i, \mathscr{A}\eta_j) = ((\eta_1, \eta_2, \cdots, \eta_{2k})X_i, (\eta_1, \eta_2, \cdots, \eta_{2k})X_j) = X_i^T S_{2k} X_j$.
Since the matrix $((\eta_i, \eta_j))_{2k\times 2k} = S_{2k} = (X_i^T S_{2k} X_j)_{2k\times 2k}$, thus we can see $A^T S_{2k} A = S_{2k}$.
(<=)
Just do the proof above reversely, completing the proof. □

We denote all the matrices of symplectic transformations under a symplectically orthonormal basis by symplectic matrices.

Definition 4.2. *Suppose V is a symplectic space with dimension $2k(2k \in \mathbb{N}_+)$ over $\mathbb{K}$, and $\eta_1, \eta_2, \cdots, \eta_{2k}$ is a symplectic basis of it. For all $\mathscr{A} \in \mathrm{SY}(V)$, the matrix under this basis is called a **symplectic matrix**. Denote $\mathrm{SY_{2k}}(\mathbb{K})$ as all the symplectic matrices over $\mathbb{K}$.*

From theorem 4.1, suppose V is a symplectic space over $\mathbb{K}$ with dimension $2k(k \in \mathbb{N}_+)$, and then we can see there is an one-to-one correspondence between $\mathrm{SY}(V)$ and $\mathrm{SY_{2k}}(\mathbb{K})$ under a symplectically orthonormal basis.

Now we try to verify there is a group structure on symplectic transformations or symplectic matrices.
Firstly verify it is closed under multiplication.

Property 4.2. *The multiplication of finite symplectic matrices is a symplectic matrix.*

Proof:
Suppose $A, B \in \mathrm{SY_{2k}}(k \in \mathbb{N}_+)$, and this implies $A^T S_{2k} A = S_{2k}$ and $B^T S_{2k} B = S_{2k}$.
$\implies (AB)^T S_{2k}(AB) = B^T(A^T S_{2k} A)B = B^T S_{2k} B = S_{2k}$.
Therefore, AB is a symplectic matrix; according to mathematical induction, the multiplication of finite symplectic matrices is a symplectic matrix, as desired. □

Secondly ensure the existence of multiplication inverses.

Property 4.3. *Symplectic matrices must be invertible.*

Proof:
We just need to considerate symplectic matrices.
Suppose $A \in \mathrm{SY_{2k}}(\mathbb{K})(k \in \mathbb{N}_+)$, and this implies $A^T S_{2k} A = S_{2k}$.
Then $\det(A^T S_{2k} A) = \det^2(A) = \det(S_{2k}) = 1 \implies \det(A) \neq 0$, which implies A is invertible. □

Remark. *Due to the facts above, as well as the associativity of the matrix multiplication, it's obvious that both of $(\mathrm{SY}(V), \circ)$ and $(\mathrm{SY_{2k}}(\mathbb{K}), \cdot)$ are groups.*

Definition 4.3. *Denote $\mathrm{SY_{2k}}(\mathbb{K})(k \in \mathbb{N}_+)$ as a 2k order **symplectic group** over $\mathbb{K}$.*

We will illustrate another way to describe a symplectic matrix by submatrices, which is more suitable for complex calculation.

Theorem 4.2. *Suppose $A \in M_{2k}(\mathbb{K})(k \in \mathbb{N}_+)$, and separate A into four submatrices i.e.*

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} (A_1, A_2, A_3, A_4 \in (\ M_k(\mathbb{K}))). \tag{14}$$

Then A is a symplectic matrix if and only if

$$\begin{aligned} &A_1^T A_3 = A_3^T A_1, A_2^T A_4 = A_4^T A_2; \\ &A_1^T A_4 - A_3^T A_2 = E. \end{aligned} \tag{15}$$

Proof:
Suppose A is a symplectic matrix, and separate it as above.
(=>)

Since it's a symplectic matrix, then $A^T S_{2k} A = S_{2k}$,
i.e.

$$\begin{pmatrix} A_1^T & A_3^T \\ A_2^T & A_4^T \end{pmatrix} \begin{pmatrix} & -E \\ E & \end{pmatrix} \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} = \begin{pmatrix} A_3^T A_1 - A_1^T A_3 & A_3^T A_2 - A_1^T A_4 \\ A_4^T A_1 - A_2^T A_3 & A_4^T A_2 - A_2^T A_4 \end{pmatrix} = \begin{pmatrix} & -E \\ E & \end{pmatrix} \tag{16}$$

Comparing the last two matrices, we get the conclusion.
(<=)
From equation (16), it's clear that this direction is true, completing the proof. □

Corollary 4.1. *Suppose $A \in SY_{2k}(\mathbb{K})(k \in \mathbb{N}_+)$, and separate A the same as equation (14). Then $A_1^T A_3$ and $A_2^T A_4$ are both symmetry matrices.*

Proof:
From theorem 4.2, $A_1^T A_3 = (A_1^T A_3)^T, A_2^T A_4 = (A_2^T A_4)^T$.
Therefore, $A_1^T A_3$ and $A_2^T A_4$ are both symmetry matrices, as desired. □

4.2 Symplectically Self-adjoint Transformations

We will define a self-adjoint linear transformation on a symplectic space.

Definition 4.4. *Suppose V is a symplectic space. Then $\mathscr{A} \in \mathrm{End}(V)$ is a symplectically self-adjoint transformations if and only if*

$$(\mathscr{A}\alpha, \beta) = (\alpha, \mathscr{A}\beta) \ (\forall \alpha, \beta \in V). \tag{17}$$

Next we will find the delicate characteristic of self-adjoint linear transformations' matrix under a symplectically orthonormal basis.

Theorem 4.3. *Suppose V is a symplectic space with dimension $2k(k \in \mathbb{N}_+)$, and $\xi_1, \xi_2, \cdots, \xi_k, \eta_1, \eta_2, \cdots, \eta_k$ is a symplectically orthonormal basis. For $\mathscr{A} \in \mathrm{End}(V)$ fulfilling $\mathscr{A}(\xi_1, \xi_2, \cdots, \xi_k, \eta_1, \eta_2, \cdots, \eta_k) = (\xi_1, \xi_2, \cdots, \xi_k, \eta_1, \eta_2, \cdots, \eta_k)A$, separate A into four submatrices*

$$A = \begin{pmatrix} A_1 & B_1 \\ B_2 & A_2 \end{pmatrix} (A_1, A_2, B_1, B_2 \in \mathrm{M_k}(\mathbb{K})). \tag{18}$$

Then $\mathscr{A}$ is a symplectically self-adjoint transformation if and only if

$$B_1^T = -B_1, B_2^T = -B_2, A_1^T = -A_2. \tag{19}$$

Proof:
(=>)
Let $A = (a_{ij})_{2k \times 2k}$, and then
$\mathscr{A}\xi_i = \sum\limits_{j=1}^{k} a_{ji}\xi_j + \sum\limits_{j=1}^{k} a_{(j+k)i}\eta_j$, and $\mathscr{A}\eta_i = \sum\limits_{j=1}^{k} a_{j(i+k)}\xi_j + \sum\limits_{j=1}^{k} a_{(j+k)(i+k)}\eta_j$,
which implies $\forall i, j \in \{1, 2, \cdots, k\}$
$(\mathscr{A}\xi_i, \xi_j) = (\sum\limits_{j=1}^{k} a_{ji}\xi_j, \xi_j) + (\sum\limits_{j=1}^{k} a_{(j+k)i}\eta_j, \xi_j) = (\sum\limits_{j=1}^{k} a_{(j+k)i}\eta_j, \xi_j) = a_{(j+k)i}$,
$(\xi_i, \mathscr{A}\xi_j) = (\xi_i, \sum\limits_{i=1}^{k} a_{ij}\xi_i) + (\xi_i, \sum\limits_{i=1}^{k} a_{(i+k)j}\eta_i) = (\xi_i, \sum\limits_{i=1}^{k} a_{(i+k)j}\eta_i) = -a_{(i+k)j}$;
$(\mathscr{A}\eta_i, \eta_j) = (\sum\limits_{j=1}^{k} a_{j(i+k)}\xi_j, \eta_j) + (\sum\limits_{j=1}^{k} a_{(j+k)(i+k)}\eta_j, \eta_j) = (\sum\limits_{j=1}^{k} a_{j(i+k)}\xi_j, \eta_j) = -a_{j(i+k)}$,
$(\eta_i, \mathscr{A}\eta_j) = (\eta_i, \sum\limits_{i=1}^{k} a_{i(j+k)}\xi_i) + (\eta_i, \sum\limits_{i=1}^{k} a_{(i+k)(j+k)}\eta_i) = (\eta_i, \sum\limits_{i=1}^{k} a_{i(j+k)}\xi_i) = a_{i(j+k)}$;
$(\mathscr{A}\xi_i, \eta_j) = (\sum\limits_{j=1}^{j} a_{ji}\xi_j, \eta_j) + (\sum\limits_{j=1}^{k} a_{(j+k)i}\eta_j, \eta_j) = (\sum\limits_{j=1}^{j} a_{ji}\xi_j, \eta_j) = -a_{ji}$,

$(\xi_i, \mathscr{A}\eta_j) = (\xi_i, \sum\limits_{i=1}^{k} a_{i(j+k)}\xi_i) + (\xi_i, \sum\limits_{i=1}^{k} a_{(i+k)(j+k)}\eta_i) = (\xi_i, \sum\limits_{i=1}^{k} a_{(i+k)(j+k)}\eta_i) = a_{(i+k)(j+k)}$.
Since $(\xi_i, \mathscr{A}\xi_j) = (\mathscr{A}\xi_i, \xi_j)$, $(\mathscr{A}\eta_i, \eta_j) = (\eta_i, \mathscr{A}\eta_j)$ and $(\mathscr{A}\xi_i, \eta_j) = (\xi_i, \mathscr{A}\eta_j)$,
we conclude that $\forall i, j \in \{1, 2, \cdots, k\}$, $a_{i(j+k)} = -a_{j(i+k)}$, $a_{(j+k)i} = -a_{(i+k)j}$, and $a_{ij} = -a_{(j+k)(i+k)}$.
Thus $B_1^T = -B_1, B_2^T = -B_2, A_1^T = -A_2$.
(<=)
Suppose $B_1^T = -B_1, B_2^T = -B_2, A_1^T = -A_2$.
According to the opposite direction's proof, we can see $\mathscr{A}$ is symplectically self-adjoint on a basis, i.e. $\forall i, j \in \{1, 2, \cdots, k\}, (\xi_i, \mathscr{A}\xi_j) = (\mathscr{A}\xi_i, \xi_j)$, $(\mathscr{A}\eta_i, \eta_j) = (\eta_i, \mathscr{A}\eta j)$ and $(\mathscr{A}\eta_i, \xi_j) = (\eta_i, \mathscr{A}\xi_j)$.
$\forall \alpha, \beta \in V$, suppose $\alpha = \sum\limits_{i=1}^{k} p_i\xi_i + \sum\limits_{i=1}^{k} p_{i+k}\eta_i$, and $\beta = \sum\limits_{i=1}^{k} q_i\xi_i + \sum\limits_{i=1}^{k} q_{i+k}\eta_i$, then

$$
\begin{aligned}
&(\mathscr{A}\alpha, \beta) = (\mathscr{A}(\sum_{i=1}^{k} p_i\xi_i + \sum_{i=1}^{k} p_{i+k}\eta_i), \sum_{i=1}^{k} q_i\xi_i + \sum_{i=1}^{k} q_{i+k}\eta_i) \\
&= (\sum_{i=1}^{k} p_i\mathscr{A}\xi_i, \sum_{i=1}^{k} q_i\xi_i) + (\sum_{i=1}^{k} p_{i+k}\mathscr{A}\eta_i, \sum_{i=1}^{k} q_i\xi_i) + (\sum_{i=1}^{k} p_i\mathscr{A}\xi_i, \sum_{i=1}^{k} q_{i+k}\eta_i) + (\sum_{i=1}^{k} p_{i+k}\mathscr{A}\eta_i, \sum_{i=1}^{k} q_{i+k}\eta_i) \\
&= (\sum_{i=1}^{k} p_i\xi_i, \sum_{i=1}^{k} q_i\mathscr{A}\xi_i) + (\sum_{i=1}^{k} p_{i+k}\eta_i, \sum_{i=1}^{k} q_i\mathscr{A}\xi_i) + (\sum_{i=1}^{k} p_i\xi_i, \sum_{i=1}^{k} q_{i+k}\mathscr{A}\eta_i) + (\sum_{i=1}^{k} p_{i+k}\eta_i, \sum_{i=1}^{k} q_{i+k}\mathscr{A}\eta_i) \\
&= (\sum_{i=1}^{k} p_i\xi_i + \sum_{i=1}^{k} p_{i+k}\eta_i, \mathscr{A}(\sum_{i=1}^{k} q_i\xi_i + \sum_{i=1}^{k} q_{i+k}\eta_i)) = (\alpha, \mathscr{A}\beta) \qquad .
\end{aligned}
$$

Thus $\mathscr{A}$ is symplectically self-adjoint for all vectors in V, completing the proof. □

In order to make theorem 4.3 more elegant, we give the following equivalent theorem.

Theorem 4.4. *Suppose V is a symplectic space with dimension $2k(k \in \mathbb{N}_+)$, and $\mathscr{A} \in \mathrm{End}(V)$. Then $\mathscr{A}$ is a symplectically self-adjoint transformation if and only if*

$$A^T S_{2k} = S_{2k} A. \tag{20}$$

Proof:
Separate A into four submatrices

$$A = \begin{pmatrix} A_1 & B_1 \\ B_2 & A_2 \end{pmatrix} (A_1, A_2, B_1, B_2 \in \mathrm{M_k}(\mathbb{K})).$$

Then calculate the LHS and RHS.

$$LHS = \begin{pmatrix} B_2^T & -A_1^T \\ A_2^T & -B_1^T \end{pmatrix},$$
$$RHS = \begin{pmatrix} -B_2 & -A_1 \\ A_1 & B_1 \end{pmatrix}.$$

From theorem 4.3, we get the conclusion by comparing LHS and RHS, as desired. □